

INSTABILITY OF A PARABOLIC EQUATION WITH A QUADRATIC NONLINEARITY

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ABSTRACT. A nonlinear parabolic differential equation with a quadratic nonlinearity is presented which has at least one equilibrium. The linearization about this equilibrium is asymptotically stable, but by using a technique inspired by H. Fujita, we show that the equilibrium is unstable in the nonlinear setting. The perturbations used have the property that they are small in every L^p norm, yet they result in solutions which fail to be global.

1. INTRODUCTION

This article demonstrates that in infinite-dimensional settings, stability of the linearization about an equilibrium of a dynamical system is not sufficient to ensure that the equilibrium is stable. This is in stark contrast to the situation in finite-dimensional settings, where stability of the linearized system implies stability of the equilibrium. (See [1], for instance.) A crucial point is that the system exhibited has a spectrum which includes zero, so stability is possible (as in the unforced heat equation), though not guaranteed.

We study classical solutions to the Cauchy problem

$$(1) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) - 2f(x)u(t, x) - u^2(t, x) \\ u(0, x) = h(x) \in C^\infty(\mathbb{R}) \\ t > 0, x \in \mathbb{R}, \end{cases}$$

where $f \in C_0^\infty(\mathbb{R})$ is a positive function with two bounded derivatives. (By C_0^∞ , we mean the space of smooth functions which decay to zero.) Since the linear portion of the right side of (1) is a sectorial operator, we can use it to define a nonlinear semigroup. [6] [10] The standard regularity theory for parabolic equations turns (1) into a smooth dynamical system, the behavior of which is largely controlled by its equilibria. This problem evidently has at least one equilibrium, namely $u(t, x) = 0$ for all t, x . Depending on the exact choice of f , there may be other equilibria, however they will not concern us here. The linearized form of (1) about this equilibrium is evidently

$$(2) \quad \frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) - 2f(x)u(t, x).$$

The zero function is asymptotically stable for the linearized problem, by a standard comparison principle argument. [3] However, using a technique pioneered by Fujita in [5], we will show that this equilibrium is not stable in the nonlinear problem, even if the initial condition has small p -norm for every $1 \leq p \leq \infty$. Fujita showed that if $f \equiv 0$, then the zero function is an unstable equilibrium of (1). The cause of the instability in (1) is the decay of f , for if $f = \text{const} > 0$, then the comparison principle shows that the zero function is stable. We extend Fujita's result, so that

roughly speaking, since $f \rightarrow 0$ away from the origin, the system is less stable to perturbations away from the origin. Another indication that there may be instability lurking (though not conclusive proof) is that the decay of f means that the spectrum of the linearized operator on the right side of (2) includes zero. [9]

2. MOTIVATION

The problem (1) describes a reaction-diffusion equation [4], or a diffusive logistic population model with a spatially-varying carrying capacity. The choice of f positive means that the equilibrium $u \equiv 0$ describes a population saturated at its carrying capacity. Without the diffusion term, this situation is well known to be stable. The decay condition on f means that the carrying capacity diminishes away from the origin.

The spatial inhomogeneity of f makes the analysis of (1) much more complicated than that of typical reaction-diffusion equations. The existence of additional equilibria for (1) is a fairly difficult problem, which depends delicately on f . (See [2] for a proof of existence of equilibria in a related setting.)

3. INSTABILITY OF THE EQUILIBRIUM

Given an $\epsilon > 0$, we will construct an initial condition $h \in C^\infty(\mathbb{R})$ for the problem (1), with $\|h\|_p < \epsilon$ for each $1 \leq p \leq \infty$, such that $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T < \infty$. In particular, this implies that $u \equiv 0$ is not a stable equilibrium of (1), at least insofar as classical solutions are concerned. We employ a technique of Fujita, which provides sufficient conditions for equations like (1) to blow up. [5] (Additionally, [3] contains a more elementary discussion of the technique with a similar construction.) Our choice for h can be thought of as a sequence of progressively shifted gaussians, and we will demonstrate that though each has smaller p -norm than the previous, the solution started at h still blows up.

3.1. The technique of Fujita. The technique of Fujita examines the blow-up behavior of nonlinear parabolic equations by treating them as ordinary differential equations on a Hilbert space. Suppose $u(t)$ solves

$$(3) \quad \frac{\partial u(t)}{\partial t} = Lu(t) + N(u(t), t),$$

where L is a linear operator not involving t , and N may be nonlinear and may depend on t . Suppose that $v(t)$ solves

$$(4) \quad \frac{\partial v(t)}{\partial t} = -L^*v(t),$$

where L^* is the adjoint of L . Let $J(t) = \langle v(t), u(t) \rangle$. We observe that if $|J(t)| \rightarrow \infty$ then either $\|v(t)\|$ or $\|u(t)\|$ also does. So if $v(t)$ does not blow up, then we can show that $\|u(t)\|$ blows up, and perhaps more is true. If we differentiate $J(t)$, we obtain the identity

$$\begin{aligned} \frac{d}{dt}J(t) &= \frac{d}{dt} \langle v(t), u(t) \rangle \\ &= \left\langle \frac{dv}{dt}, u(t) \right\rangle + \left\langle v(t), \frac{du}{dt} \right\rangle \\ &= \langle -L^*v(t), u(t) \rangle + \langle v(t), Lu(t) + N(u(t), t) \rangle \\ &= \langle v(t), N(u(t), t) \rangle, \end{aligned}$$

where there is typically a technical justification required for the second equality. It is often possible to find a bound for $\langle v(t), N(u(t), t) \rangle$ in terms of $J(t)$. So then the method provides a fence (in the sense of [7]) for $J(t)$, which we can solve to give a bound on $|J(t)|$. As a result, the blow-up behavior of $u(t)$ is controlled by the solution of an *ordinary* differential equation (for $J(t)$) and a *linear* parabolic equation (for $v(t)$), both of which are much easier to examine than the original nonlinear parabolic equation.

3.2. Instability in L^p for $1 \leq p \leq \infty$. We begin our application of the method of Fujita by working with $L = \frac{\partial^2}{\partial x^2} - 2f$ and $N(u) = -u^2$ in (3). Since (4) is then not well-posed for all t , we must be a little more careful than the method initially suggests. For this reason, we consider a family of solutions v_ϵ to (4) that have slightly extended domains of definition. It will also be important, for technical reasons, to enforce the assumption that the first and second derivatives of f are bounded.

Definition 1. Suppose $w = w(t, x)$ solves

$$(5) \quad \begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - 2f(x)w(t, x) \\ w(0, x) = w_0(x) \geq 0. \end{cases}$$

Define $v_\epsilon(s, x) = w(t - s + \epsilon, x)$ for fixed $t > 0$ and $s < t + \epsilon$. Notice that by the comparison principle, $v_\epsilon(s, x) \geq 0$.

Lemma 2. Suppose that w solves (5). Then $w, \frac{\partial w}{\partial x} \in C_0(\mathbb{R})$.

Proof. The standard existence and regularity theorems for linear parabolic equations (see [10], for example) give that $w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2} \in L^2(\mathbb{R})$ and that $w \in C^2(\mathbb{R})$. The comparison principle, applied to $\frac{\partial}{\partial t} \frac{\partial w}{\partial x}$ and $\frac{\partial}{\partial t} \frac{\partial^2 w}{\partial x^2}$ gives that the first and second derivatives of w are bounded for each fixed t . (This uses our assumption that f has two bounded derivatives.)

The lemma follows from a more general result: if $g \in C^1 \cap L^p(\mathbb{R})$ for $1 \leq p < \infty$ and $g' \in L^\infty(\mathbb{R})$, then $g \in C_0(\mathbb{R})$. To show this, we suppose the contrary, that $\lim_{x \rightarrow \infty} g(x) \neq 0$ (and possibly doesn't exist). By definition, this implies that there is an $\epsilon > 0$ such that for all $x > 0$, there is a y satisfying $y > x$ and $|g(y)| > \epsilon$. Let $S = \{y \mid |g(y)| > \epsilon\}$, which is a union of open intervals, is of finite measure, and has $\sup S = \infty$. Let $T = \{y \mid |g(y)| > \epsilon/2\}$. Note that T contains S , but since g' is bounded, for each $x \in S$, there is a neighborhood of x contained in T of measure at least $\epsilon/\|g'\|_\infty$. Hence, since $\sup T = \sup S = \infty$, T cannot be of finite measure, which contradicts the fact that $g \in L^p(\mathbb{R})$ with $1 \leq p < \infty$. \square

Lemma 3. Suppose $u : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is a classical solution to (1) with $u \leq 0$ and $u(t) \in L^\infty(\mathbb{R})$ for each $t \in [0, T)$. Then

$$(6) \quad - \int w(t, x) h(x) dx \leq \left(\int_0^t \frac{1}{\|w(s)\|_1} ds \right)^{-1},$$

where w is defined as in Definition 1.

Proof. Define

$$(7) \quad J_\epsilon(s) = \int v_\epsilon(s, x) u(s, x) dx.$$

First of all, we observe that since $u \in L^\infty(\mathbb{R})$, $v_\epsilon(s, \cdot)u(s, \cdot)$ is in $L^1(\mathbb{R})$ for each $s < t$.

Now suppose we have a sequence $\{m_n\}$ of compactly supported smooth functions with the following properties: [8]

- $m_n \in C^\infty(\mathbb{R})$,
- $m_n(x) \geq 0$ for all x ,
- $\text{supp}(m_n)$ is contained in the interval $(-n-1, n+1)$, and
- $m_n(x) = 1$ for $|x| \leq n$.

Then it follows that

$$J_\epsilon(s) = \lim_{n \rightarrow \infty} \int v_\epsilon(s, x)u(s, x)m_n(x)dx.$$

Now

$$\begin{aligned} \frac{d}{ds} J_\epsilon(s) &= \frac{d}{ds} \lim_{n \rightarrow \infty} \int v_\epsilon(s, x)u(s, x)m_n(x)dx \\ &= \lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x)) m_n(x)dx. \end{aligned}$$

We'd like to exchange limits using uniform convergence. To do this we show that

$$(8) \quad \lim_{n \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x)) m_n(x)dx$$

exists and the inner limit is uniform. We show both together by a little computation, using uniform convergence and LDCT:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \lim_{h \rightarrow 0} \frac{1}{h} \int (v_\epsilon(s+h, x)u(s+h, x) - v_\epsilon(s, x)u(s, x)) m_n(x)dx \\ &= \lim_{n \rightarrow \infty} \int \left(\frac{\partial}{\partial s} v_\epsilon(s, x)u(s, x) + v_\epsilon(s, x) \frac{\partial}{\partial s} u(s, x) \right) m_n(x)dx \\ &= \lim_{n \rightarrow \infty} \int \left(-\frac{\partial^2}{\partial x^2} v_\epsilon(s, x) + 2f(x)v_\epsilon(s, x) \right) u(s, x)m_n(x) + \\ & \quad v_\epsilon(s, x) \left(\frac{\partial^2}{\partial x^2} u(s, x) - u^2(s, x) - 2f(x)u(s, x) \right) m_n(x)dx \\ &= \lim_{n \rightarrow \infty} \int -v_\epsilon(s, x)u^2(s, x)m_n(x)dx. \end{aligned}$$

Minkowski's inequality has that

$$\left| \int v_\epsilon u m_n dx \right| \leq \int v_\epsilon |u| m_n dx \leq \left(\int v_\epsilon m_n dx \right)^{1/2} \left(\int v_\epsilon u^2 m_n dx \right)^{1/2},$$

since $v_\epsilon, m_n \geq 0$. This gives that

$$\begin{aligned}
 & \int -v_\epsilon(s, x)u^2(s, x)m_n(x)dx \\
 & \leq -\frac{(\int v_\epsilon u m_n dx)^2}{\int v_\epsilon m_n dx} \\
 & \leq -\frac{(\int v_\epsilon u dx)^2}{\int v_\epsilon m_1 dx},
 \end{aligned}$$

hence the inner limit of (8) is uniform. On the other hand,

$$|v_\epsilon(s, x)u^2(s, x)m_n(x)| \leq v_\epsilon(s, x)\|u(s)\|_\infty^2 \in L^1(\mathbb{R})$$

so the double limit of (8) exists by dominated convergence. Thus we have the fence

$$(9) \quad \frac{dJ_\epsilon(s)}{ds} \leq -\frac{(J_\epsilon(s))^2}{\|v_\epsilon(s)\|_1}.$$

We solve the fence (9) to obtain (note $J_\epsilon \leq 0$)

$$\begin{aligned}
 \frac{1}{\|v_\epsilon(s)\|_1} & \leq -\frac{dJ_\epsilon(s)}{ds} \frac{1}{(J_\epsilon(s))^2} \\
 \int_0^t \frac{1}{\|v_\epsilon(s)\|_1} ds & \leq \frac{1}{J_\epsilon(t)} - \frac{1}{J_\epsilon(0)} \\
 \int_0^t \frac{1}{\|v_\epsilon(s)\|_1} ds & \leq -\frac{1}{J_\epsilon(0)}.
 \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ of both sides of the inequality yields

$$-\int w(t, x)h(x)dx \leq \left(\int_0^t \frac{1}{\|w(t-s)\|_1} ds\right)^{-1} = \left(\int_0^t \frac{1}{\|w(s)\|_1} ds\right)^{-1},$$

as desired. \square

Remark 4. Since we are interested in proving the instability of the zero function in (1), consider $u(0, x) = h(x) = -\epsilon$ for $\epsilon > 0$. Then (6) takes on the simple form

$$(10) \quad \epsilon \int_0^t \frac{\|w(t)\|_1}{\|w(s)\|_1} ds \leq 1.$$

So in particular, $\|u(t)\|_\infty$ blows up if there exists a $T > 0$ such that $\epsilon \int_0^T \frac{\|w(T)\|_1}{\|w(s)\|_1} ds > 1$.

The stability of the zero function in (1) depends on the stability of the zero function in (5) – the linearized problem. If the zero function in the linearized problem is very strongly attractive, say $\|w(t)\|_1 \sim e^{-t}$, then

$$\int_0^t \frac{e^{-t}}{e^{-s}} ds = (1 - e^{-t}) < 1,$$

and so a small choice of $\epsilon < 1$ does not cause blow-up via a violation of (10). On the other hand, blow-up occurs if it is less attractive, say $\|w(t)\|_1 \sim t^{-\alpha}$ for $\alpha \geq 0$. Because then

$$\int_0^t \frac{s^\alpha}{t^\alpha} ds = \frac{t}{\alpha + 1},$$

whence blow-up occurs before $t = \frac{\alpha+1}{\epsilon}$.

In the particular case of $f(x) = 0$ for all x , we note that w is simply a solution to the heat equation, which has $\|w(t)\|_1 = \|w_0\|_1$ for all t (by direct computation using the fundamental solution, say), so blow up occurs. Thus we can recover a special case of the original blow-up result of Fujita in [5].

Theorem 5. *Suppose a sufficiently small $\epsilon > 0$ is given. Then for a certain choice of initial condition $h(x)$ with $\|h\|_p < \epsilon$ for all $1 \leq p \leq \infty$, there exists a $T > 0$ for which $\lim_{t \rightarrow T^-} \|u(t)\|_\infty = \infty$.*

Proof. First, it suffices to choose $\|u(0)\|_1 < \epsilon$ and $\|u(0)\|_\infty < \epsilon$, since

$$\|u\|_p = \left(\int |u|^p dx \right)^{1/p} \leq \|u\|_\infty^{(p-1)/p} \|u\|_1^{1/p} < \epsilon.$$

We assume, contrary to what is to be proven, that $\|u(t)\|_\infty$ does not blow up for any finite t . In other words, assume that $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a classical solution to (1), with $\|u(t)\|_\infty < \infty$ for all t . We make several definitions:

- Choose $0 < \beta < \min \left\{ \epsilon, \frac{\epsilon^4}{16\pi^2} \right\}$.
- Choose $\gamma > 0$ small enough so that

$$(11) \quad \frac{\beta}{27\gamma^2} = K,$$

for some some arbitrary $K > 1$.

- Since $0 \leq f \in C_0^\infty(\mathbb{R})$, we can choose an x_1 such that

$$(12) \quad f(x) \leq \gamma \text{ when } x < x_1.$$

- Next, we choose $x_0 < x_1$ so that

$$(13) \quad \sqrt{t} \|f\|_\infty \left(1 - \operatorname{erf} \left(\frac{x_1 - x_0}{2\sqrt{t}} \right) \right) < \gamma$$

for all $0 < t < \frac{1}{4\gamma^2}$. Notice that any choice less than x_0 will also work.

- Choose the initial condition for (1) to be

$$(14) \quad u(0, x) = h(x) = -\beta e^{\beta^{3/2}(x-x_0)^2}.$$

This choice of initial condition has $\|u(0)\|_\infty = \beta < \epsilon$, $\|u(0)\|_1 = 2\pi^{1/2}\beta^{1/4} < \epsilon$, and $\left\| \frac{\partial^2 u(0)}{\partial x^2} \right\|_\infty = \mu = 2\beta^{5/2}$. (The value of μ will be important shortly.)

- Finally, let $w_0(y) = \delta(y - x_0)$ (the Dirac δ -distribution), and suppose that w solves (5). In other words, choose w to be the fundamental solution to (5) concentrated at x_0 . Note that the maximum principle ensures both that $w(t, x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$ and that $\|w(t)\|_1 \leq \|w(0)\|_1 = 1$ for all $t > 0$. This allows us to rewrite (6) as

$$(15) \quad -t \int w(t, x) h(x) dx \leq 1.$$

Now we estimate the integral in (15). Notice that

$$\begin{aligned} \frac{d}{dt} \int w(t, x) (-h(x)) dx &= \int \left(\frac{\partial^2 w}{\partial x^2} - 2f(x)w(t, x) \right) (-h(x)) dx \\ &= \int \left(-\frac{\partial^2 u}{\partial x^2} + 2f(x)h(x) \right) w(t, x) dx, \end{aligned}$$

where Lemma 2 eliminates the boundary terms. Now suppose z solves the heat equation with the same initial condition as w , namely

$$(16) \quad \begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} \\ z(0, x) = w_0(x) = \delta(x - x_0). \end{cases}$$

The comparison principle establishes that $z(t, x) \geq w(t, x)$ for all $t > 0$ and $x \in \mathbb{R}$, since $f, w \geq 0$. As a result, we have that

$$\begin{aligned} \frac{d}{dt} \int w(t, x) (-h(x)) dx &\geq \int \left(-\left| \frac{\partial^2 u}{\partial x^2} \right| + 2f(x)h(x) \right) z(t, x) dx \\ &\geq -\mu - 2\beta \int f(x) z(t, x) dx, \end{aligned}$$

where $\mu = \left\| \frac{\partial^2 u}{\partial x^2}(0) \right\|_\infty$ and $\beta = \|u(0)\|_\infty$, which is an integrable equation. As a result,

$$(17) \quad \int w(t, x) (-h(x)) dx \geq \beta - \mu t - 2\beta \int_0^t \int \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-y)^2}{4s}} w_0(y) dy dx ds.$$

On the other hand using our choice for w_0 ,

$$\begin{aligned} \int_0^t \int \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-y)^2}{4s}} w_0(y) dy dx ds &= \int_0^t \int f(x) \frac{1}{\sqrt{4\pi s}} e^{-\frac{(x-x_0)^2}{4s}} dx ds \\ &\leq \int_0^t \frac{1}{\sqrt{4\pi s}} \left(\gamma \int_{-\infty}^{x_1} e^{-\frac{(x-x_0)^2}{4s}} dx + \|f\|_\infty \int_{x_1}^\infty e^{-\frac{(x-x_0)^2}{4s}} dx \right) ds \\ &\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} \|f\|_\infty \int_0^t 1 - \operatorname{erf}\left(\frac{x_1 - x_0}{2\sqrt{s}}\right) ds \\ &\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} \|f\|_\infty \int_0^t 1 - \operatorname{erf}\left(\frac{x_1 - x_0}{2\sqrt{t}}\right) ds \\ &\leq \frac{\gamma\sqrt{t}}{4} + \frac{1}{2} t \|f\|_\infty \left(1 - \operatorname{erf}\left(\frac{x_1 - x_0}{2\sqrt{t}}\right) \right) \\ &\leq \frac{3\gamma\sqrt{t}}{4} \leq \gamma\sqrt{t}, \end{aligned}$$

we have used (12), (13), and assumed that $0 < t < \frac{1}{4\gamma^2}$. Then (15) becomes

$$1 \geq t \int w(t, x) (-h(x)) dx \geq \beta t - \mu t^2 - 2\beta\gamma t\sqrt{t} = -2\beta^{5/2}t^2 - \frac{2\beta^{3/2}t^{3/2}}{\sqrt{27K}} + \beta t,$$

using our choices of μ, γ , and initial condition. Maple reports that the maximum of $A(t) = -2\beta^{5/2}t^2 - \frac{2\beta^{3/2}t^{3/2}}{\sqrt{27K}} + \beta t$ is unique, occurs at $0 < t_0 < \frac{1}{4\gamma^2}$, and has the asymptotic expansion

$$A(t_0) \sim K - 18K\sqrt{\beta} + 432K^3\beta + O(\beta^{3/2}).$$

Thus for all small enough $\epsilon > \beta$, we obtain a contradiction to (15) since $K > 1$. Thus, for some $T < t_0 < \infty$, $\lim_{t \rightarrow T^-} \|u(t)\|_\infty = \infty$. \square

4. DISCUSSION

Theorem 5 gives a fairly strong instability result. No matter how small an initial condition to (1) is chosen, even with all p -norms chosen small, solutions can blow up so quickly that they fail to exist for all t . This precludes any kind of stability for classical solutions. Like the analogous result in Fujita's paper, the kind of initial conditions which can be responsible for blow up are of the nicest kind imaginable – gaussians in either case!

It must be understood that the argument in Theorem 5 depends crucially on the decay of f . Without it, the lower bound on $\int w(t, x)(-h(x))dx$ decreases too quickly. Indeed, if $f = \text{const} > 0$ and $h(x) > -f$, then the comparison principle demonstrates that the zero function is asymptotically stable. On the other hand, any rate of decay for f satisfies the hypotheses of Theorem 5, and so will cause (1) to exhibit instability.

Finally, although we have examined the case where the nonlinearity in (1) is due to u^2 , there is no obstruction to extending the analysis to any nonlinearity like $|u|^k$, with degree k greater than 2. A higher-degree nonlinearity would result in a somewhat different form for (6), but this presents no further difficulties to the argument. Indeed, by analogy with Fujita's work, higher-degree nonlinearities would result in significantly faster blow-up.

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